

SPHERICAL POSETS FROM COMMUTING ELEMENTS

CİHAN OKAY

ABSTRACT. In this paper we study the poset of cosets of abelian subgroups in an extraspecial p -group. We prove that the universal cover of the nerve of this poset is homotopy equivalent to a wedge of n -spheres where $2n$ is the rank of its Frattini quotient.

1. INTRODUCTION

Collections of subgroups of a group G play a central role in topology and representation theory. Ordered under the inclusion relation such collections are partially ordered sets. Taking the nerve gives rise to natural examples of G -spaces. Topological properties of such spaces reflect the structure of the group. Tits building associated to an algebraic group is one of the classical examples. By the Solomon–Tits theorem [11] we know that the resulting space has the homotopy type of a wedge of spheres. Its top dimensional homology affords the Steinberg representation of the group. In [10] Quillen showed that for finite Chevalley groups in characteristic p the Tits building is homotopy equivalent to the poset of non-trivial elementary abelian p -subgroups. Lusztig studied an affine version of the Tits building [7] for the general linear group over a finite field. In this case the collection of proper affine subspaces $\{v + A \subset V\}$ replaces the Tits building. It turns out that this space is also a wedge of spheres. A natural generalization of this construction to finite groups is studied by Brown in [4]. When G is solvable the corresponding space of the poset $\{gA \subset G\}$ which consists of cosets of proper subgroups is a wedge of spheres. Spherical posets also play a role in homological stability results. In [12] Vogtmann showed a stability result for the orthogonal group by considering the action on the poset of singular subspaces.

The motivation behind this work was initiated by the work on a certain filtration $\{B(q, G)\}_{q \geq 2}$ of the classifying space BG introduced in [1]. There are principal G -bundles $E(q, G) \rightarrow B(q, G)$ for all $q \geq 2$ pulled back from the usual universal bundle $EG \rightarrow BG$. When $q = 2$ the space $E(2, G)$ is weakly equivalent to the nerve of the poset $O_G \mathcal{A}(G) = \{gA \subset G \mid A \text{ is an abelian subgroup}\}$. Unlike EG the space $E(2, G)$ is not contractible hence $B(2, G)$ is not an Eilenberg–MacLane space in general, as shown in [8, 9] for extraspecial p -groups. In this paper we determine the homotopy type of its universal cover. An extraspecial group E is a central extension of a cyclic group of prime order p with an elementary abelian p -group which coincides with the Frattini quotient of E .

Theorem 1.1. *Let E be an extraspecial p -group. The universal cover of $O_E \mathcal{A}(E)$ has the homotopy type of a wedge of n -spheres where $2n$ is the rank of its Frattini quotient.*

Our method is a generalization of the approach in [12]. We consider the poset $O_G \mathcal{F}$ of cosets of an arbitrary collection \mathcal{F} of subgroups. Given a normal subgroup $H \subset G$ we show how to

decompose this space with respect to H . This decomposition is a description of the space $O_G\mathcal{F}$ as a homotopy colimit. In certain cases we construct a homotopy section to show that the associated Mayer–Vietoris sequence splits. As an application we specialize to the collection of abelian subgroups. Another useful tool is a spectral sequence described in [10]. Using this spectral sequence we study the natural map $O_G\mathcal{F} \rightarrow \mathcal{F}$ defined by $gA \mapsto A$.

The organization of the paper is as follows. In §2 we describe the decomposition Theorem 2.5 and an application of the spectral sequence Theorem 2.8. In §3 we use the techniques developed in the previous section to prove our main result Theorem 1.1 (Theorem 3.1).

2. COSETS OF SUBGROUP COLLECTIONS

In this section we will study posets coming from collection of subgroups. We will not distinguish the poset \mathcal{X} from its nerve $|\mathcal{X}|$ when talking about topological properties. For basic terminology and other applications of homotopy colimits we refer to [5, Chapter slowroman-capi@]. We will be using results from [10] about posets. We start reviewing some of the basic results proved there.

Homotopy theory of posets. If $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ are two poset maps such that $f(x) \leq g(x)$ for all $x \in \mathcal{X}$ then the maps are homotopic $f \simeq g$. This homotopy property implies that a poset with an initial (or terminal) object is contractible. The fiber of a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ over $y \in \mathcal{Y}$ is defined by

$$f_{\leq y} = \{x \in \mathcal{X} \mid f(x) \leq y\}, \text{ and dually } f_{\geq y} = \{x \in \mathcal{X} \mid f(x) \geq y\}.$$

If f is the identity map we simply write $\mathcal{X}_{\geq x}$ for $f_{\geq x}$. Other variations are $\mathcal{X}_{>x}$, $\mathcal{X}_{\leq x}$, and $\mathcal{X}_{<x}$. It is a useful fact to observe that if all the fibers $f_{\geq y}$ (or $f_{\leq y}$) are contractible then f is a homotopy equivalence. Given two posets \mathcal{X} and \mathcal{Y} the join $\mathcal{X} * \mathcal{Y}$ is the disjoint union $\mathcal{X} \coprod \mathcal{Y}$ equipped with the ordering which agrees with the given one on \mathcal{X} and \mathcal{Y} and $x \leq y$ for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. The suspension $\Sigma\mathcal{X}$ is defined to be the join $\{0, 1\} * \mathcal{X}$.

Collections of subgroups. Let G be a finite group and \mathcal{F} be a collection of subgroups of G . Let $H \subset G$ be a subgroup. Given a (left) coset $gH \subset G$ we define

$$O_{gH}\mathcal{F} = \{xA \mid A \in \mathcal{F} \wedge x \in gH\}.$$

$\mathcal{F}_{\cap H}$ will denote the collection of subgroups $A \cap H$ where $A \in \mathcal{F}$. There is an H -equivariant map of posets

$$i_H : O_H\mathcal{F} \rightarrow O_H\mathcal{F}_{\cap H}$$

defined by $hA \mapsto h(A \cap H)$. Under suitable conditions this map is a weak equivalence.

Definition 2.1. We say \mathcal{F} is H -stable if for all $A \in \mathcal{F}$ the intersection $A \cap H$ is also in \mathcal{F} .

For example the collection of abelian subgroups satisfy this property with respect to any subgroup.

Proposition 2.2. *If \mathcal{F} is H -stable then $i_H : O_H\mathcal{F} \rightarrow O_H\mathcal{F}_{\cap H}$ is a weak H -equivalence equivalence.*

Proof. Note that by the assumption $\mathcal{F}_{\cap H} \subset \mathcal{F}$. Let $K \subset H$ and consider the restriction of i_H to the fixed points $i_H : (O_H \mathcal{F})^K \rightarrow (O_H \mathcal{F}_{\cap H})^K$. If the fixed points of the latter is not empty then in the fiber

$$(i_H)_{\geq hB} = \{hA \in (O_H \mathcal{F})^K \mid A \cap H \supset B\} = \{hA \in O_H \mathcal{F} \mid A \cap H \supset B \wedge K \subset hAh^{-1}\}$$

hB is initial. Hence the fiber is contractible. \square

There is a commutative diagram

$$\begin{array}{ccc} G \times_H O_H \mathcal{F} & \xrightarrow{\sim} & G \times_H O_H \mathcal{F}_{\cap H} \\ \uparrow & & \uparrow \\ O_{gH} \mathcal{F} & \xrightarrow{i_{gH}} & O_{gH} \mathcal{F}_{\cap H} \end{array} \quad (2.2.1)$$

where $i_{gH}(ghA) = gh(A \cap H)$, and the top map is a weak G -equivalence if \mathcal{F} is H -stable. Note that multiplication by an element g induces a map $O_H \mathcal{F} \xrightarrow{g} O_{gH} \mathcal{F}$ which is an isomorphism of posets.

Decomposing coset posets. We describe a decomposition of $O_G \mathcal{F}$ as a homotopy colimit. For basic properties of homotopy colimits we refer to [3, 6]. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map of posets. Let $f_{\geq -} : \mathcal{Y}^{\text{op}} \rightarrow \mathbf{S}$ denote the functor which sends an object y to the fiber $f_{\geq y}$. There is a natural map

$$\tilde{f} : \text{hocolim } f_{\geq -} \rightarrow \mathcal{X} \quad (2.2.2)$$

induced by the inclusions $f_{\geq y} \subset \mathcal{X}$. This map is a weak equivalence [6, slowromancapiv@]. In fact a similar property holds for a functor between small categories. In the case of posets one can argue as follows. The homotopy colimit is weakly equivalent to the nerve of the Grothendieck construction $\text{Gr}(f_{\geq -})$. The objects of the category $\text{Gr}(f_{\geq -})$ are pairs (y, x) such that there exists a morphism $y \rightarrow f(x)$ in \mathcal{Y} , and its morphisms $(y, x) \rightarrow (y', x')$ are given by a pair of morphisms $y \rightarrow y'$ in \mathcal{Y} and $x \rightarrow x'$ in \mathcal{X} . In particular it is a poset. Replacing the homotopy colimit in 2.2.2 by the Grothendieck construction we get a map

$$\tilde{f} : \text{Gr}(f_{\geq -}) \rightarrow \mathcal{X}$$

defined by $(y, x) \mapsto x$. The fiber $\tilde{f}_{\leq x}$ consists of pairs (y', x') with $y' \rightarrow f(x')$ and $x' \rightarrow x$. The object $(f(x), x)$ is terminal in $\tilde{f}_{\leq x}$ since

$$(y', x') \rightarrow (f(x'), x') \rightarrow (f(x), x).$$

In particular every fiber $\tilde{f}_{\leq x}$ is contractible. Therefore 2.2.2 is a weak equivalence.

Before we apply this to the poset $O_G \mathcal{F}$ we need a definition. Given a group homomorphism $f : G \rightarrow G'$ let $f(\mathcal{F})$ denote the collection which consists of the images $f(A)$ of $A \in \mathcal{F}$. The induced map $O_G \mathcal{F} \rightarrow O_{G'} f(\mathcal{F})$ will also be denoted by f . Now let H be a normal subgroup of G , and $\pi : G \rightarrow G/H$ denote the natural projection. Consider the induced map of posets

$$\pi : O_G \mathcal{F} \rightarrow O_{G/H} \pi(\mathcal{F})$$

defined by $gA \mapsto \pi(gA)$. There is a weak equivalence

$$\tilde{\pi} : \operatorname{hocolim} \pi_{\geq -} \rightarrow O_G \mathcal{F} \quad (2.2.3)$$

obtained by applying 2.2.2 to the map π .

Let $\mathcal{L}(G)$ denote the poset of all subgroups of G . An important special case is when the quotient G/H is a cyclic group C_p of prime order. We have

$$G = \coprod_{t=0}^{p-1} g^t H$$

for some $g \notin H$. If \mathcal{F} contains a non-trivial group then $O_{G/H} \pi(\mathcal{F})$ coincides with $O_{C_p} \mathcal{L}(C_p)$. In this case there are two types of fibers $\pi_{\geq G/H}$ and $\pi_{\geq g^t}$ where $1 \leq t < p$. This decomposition will be our basic tool in studying the homotopy type of $O_G \mathcal{F}$. Let \mathcal{F}^H denote the poset $\{A \in \mathcal{F} \mid AH = G\}$ that is the collection of complements of H .

Proposition 2.3. *Let $0 \rightarrow H \rightarrow G \xrightarrow{\pi} C_p \rightarrow 0$ be an exact sequence of groups where p is a prime. Assume that \mathcal{F} contains a non-trivial group. Then there is a homotopy equivalence*

$$\tilde{\pi} : \operatorname{hocolim}_{O_{C_p} \mathcal{L}(C_p)^{op}} \pi_{\geq -} \rightarrow O_G \mathcal{F}$$

where the functor $\pi_{\geq -}$ is given by

$$\pi_{\geq G/H} = O_G \mathcal{F}^H \quad \text{and} \quad \pi_{\geq g^t} = O_{g^t H} \mathcal{F}. \quad (2.3.1)$$

Example 2.4. Let V be a vector space over \mathbb{F}_p , and $\mathcal{T}(V)$ denote the poset of proper subspaces. Choose a subspace W of codimension one. Note that by 2.2 there is a weak equivalence

$$O_W \mathcal{T} \xrightarrow{\sim} O_W \mathcal{T}_{\cap W} \simeq \text{pt}$$

since $\mathcal{T}_{\cap W}$ contains W as a terminal object. Therefore the decomposition in 2.3 gives

$$O_V \mathcal{T} \simeq \vee^{p-1} \Sigma(O_V \mathcal{T}^W).$$

Homotopy sections. The decomposition in 2.3 gives a long exact sequence in homology groups. If we have $X = \cup_{j=1}^m X_j$ such that $X_a \cap X_b = \cap_{j=1}^m X_j$ for $a \neq b$ then there is a Mayer–Vietoris sequence

$$\cdots \rightarrow H_i(\cap_j X_j)^{\oplus m-1} \rightarrow \oplus_j H_i(X_j) \rightarrow H_i(X) \rightarrow H_{i-1}(\cap_j X_j)^{\oplus m-1} \rightarrow \cdots$$

This sequence breaks up into short exact sequences if there exist homotopy sections s_b of the inclusions $i_b : \cap_j X_j \rightarrow X_b$ satisfying certain conditions which we describe next.

Throughout we will denote the natural inclusions $O_G \mathcal{F}^H \subset O_{g^t H} \mathcal{F}$ by

$$\theta_{g^t H} : O_G \mathcal{F}^H \rightarrow O_{g^t H} \mathcal{F}$$

where $0 \leq t < p$. The restriction of θ_H to \mathcal{F}^H regarded as a sub-poset of $O_G \mathcal{F}^H$ will be denoted by the same symbol.

Theorem 2.5. Let $H \triangleleft G$ such that the quotient G/H is cyclic of prime order p . Assume that there is a map $s : O_H \mathcal{F} \rightarrow \mathcal{F}^H$ such that $\theta_H g^{-1} s \simeq \text{Id}$ and $\theta_H s \simeq 0$. Then there is a split exact sequence

$$0 \rightarrow H_i(O_G \mathcal{F}) \rightarrow H_{i-1}(O_G \mathcal{F}^H)^{\oplus(p-1)} \xrightarrow{\theta} \bigoplus_{k=1}^p H_{i-1}(O_{g^k H} \mathcal{F}) \rightarrow 0, \quad i \geq 1$$

where $\theta(x_t) = \theta_{g^{t-1}H}(x_t) - \theta_{g^t H}(x_t)$ for $1 \leq t < p$.

Proof. The Mayer-Vietoris sequence associated to the decomposition 2.2.3 gives a long exact sequence. We will show that this sequence breaks up into short exact sequences by proving that θ is surjective. For this we will construct a homotopy section. Let us denote $O_{g^k H} \mathcal{F}$ simply by $O_{g^k H}$. The conjugates of the section s makes the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & O_{gH} & \longrightarrow & O_H & \longrightarrow & O_{g^{-1}H} \longrightarrow \cdots \\ & & \theta_{gH} \uparrow & & \theta_H \uparrow & & \theta_{g^{-1}H} \uparrow \\ \cdots & \longrightarrow & g\mathcal{F}^H & \longrightarrow & \mathcal{F}^H & \longrightarrow & g^{-1}\mathcal{F}^H \longrightarrow \cdots \\ & & \downarrow \theta_{g^2 H} & \nearrow gsg^{-1} & \downarrow \theta_{gH} & \nearrow s & \downarrow \theta_H \\ \cdots & \longrightarrow & O_{g^2 H} & \longrightarrow & O_{gH} & \longrightarrow & O_H \longrightarrow \cdots \end{array} \quad (2.5.1)$$

commute. Here the horizontal maps are induced by multiplication by g^{-1} . Define the maps

$$s_{g^k H} = g^{k-1} s g^{-k} : O_{g^k H} \rightarrow g^{k-1} \mathcal{F}_H$$

for $0 \leq k < p$. Then from the diagram 2.5.1 we see that

$$\theta_{g^l H} s_{g^k H} = g^l \theta_H g^{k-1-l} s g^{-k} \simeq \begin{cases} \text{Id} & \text{if } l = k \\ 0 & \text{if } l = k - 1. \end{cases}$$

This gives a sequence of sections

$$\begin{array}{ccccc} & & O_G \mathcal{F} & \xleftarrow{s_H} & O_G \mathcal{F} \\ & \swarrow \theta_{g^{-1}H} & \searrow \theta_H & \swarrow \theta_H & \searrow \theta_{gH} \\ \cdots O_{g^{-1}H} & & O_H & & O_{gH} \cdots \end{array}$$

Let y_i be an element in the i -th factor of $H_{i-1}(O_H) \oplus \cdots \oplus H_{i-1}(O_{g^{p-1}H})$. If $i > 1$ then $s_{g^i H}(y_i)$ is the element which maps to y_i . If $i = 1$ then the $(p-1)$ -tuple $(s_H(y_1), s_H(y_1), \dots, s_H(y_1))$ maps to y_1 . \square

A poset is said to be n -spherical if it (or rather its nerve) is $(n-1)$ -connected. As an immediate application we will consider spherical posets.

Corollary 2.6. Assume that the map s exists as in 2.5. If $O_G \mathcal{F}^H$ and $O_H \mathcal{F}$ are $(n-1)$ -spherical then $O_G \mathcal{F}$ is n -spherical.

Proof. If $n = 1$ the space $X = O_G \mathcal{F}$ is connected and by 2.5 it is a wedge of circles. If $n \geq 3$ then by van Kampen's theorem X is simply connected. Now consider the case $n = 2$. Again by van Kampen's theorem there is an exact sequence

$$\prod_{k=1}^{p-1} \pi_1(O_G \mathcal{F}^H) \xrightarrow{\theta'} \prod_{k=1}^p \pi_1(O_{g^k H} \mathcal{F}) \rightarrow \pi_1(O_G \mathcal{F}) \rightarrow 1$$

where θ' is the induced map in homotopy groups. Similar to the proof of 2.5 the existence of the section s also implies that θ' is surjective. \square

A spectral sequence. In [10, Theorem 9.1] Quillen describes a spectral sequence associated to a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ of posets. He uses this spectral sequence to study spherical posets. We will describe this theorem in a slightly more general form where the fibers are not required to be connected. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map of finite posets, and $F : \mathcal{X} \rightarrow \mathbf{Ab}$ a functor. There is an induced functor $f_* : [\mathcal{Y}, \mathbf{Ab}] \rightarrow [\mathcal{X}, \mathbf{Ab}]$ between the functor categories defined by pre-composition. It has a left adjoint $f^* : [\mathcal{X}, \mathbf{Ab}] \rightarrow [\mathcal{Y}, \mathbf{Ab}]$. The Grothendieck spectral sequence associated to the composition

$$\operatorname{colim} F \cong \operatorname{colim} f^* F$$

has an E_2 -page which consists of the (left) derived functors of $f^* F$

$$E_{p,q}^2 = \operatorname{colim}_p L_q(f^* F) \Rightarrow \operatorname{colim}_{p+q} F. \quad (2.6.1)$$

The left adjoint f^* can be described explicitly. There is a pull-back diagram of posets

$$\begin{array}{ccc} f_{\leq y} & \xrightarrow{i_y} & \mathcal{X} \\ \downarrow & & \downarrow f \\ \mathcal{Y}_{\leq y} & \longrightarrow & \mathcal{Y} \end{array}$$

and the functor $f^* F : \mathcal{Y} \rightarrow \mathbf{Ab}$ can be defined by $f^* F(y) = \operatorname{colim} (i_y)_* F$. Instead of working with $f_{\leq y}$ it is more convenient for us to use $f_{\geq y}$. One can pass from one to the other by considering the opposite map $f^{\operatorname{op}} : \mathcal{X}^{\operatorname{op}} \rightarrow \mathcal{Y}^{\operatorname{op}}$.

Let us take the constant functor $\mathbb{Z} : \mathcal{X} \rightarrow \mathbf{Ab}$ to compute the homology of \mathcal{X} . Assume that for all y the fibers $f_{\geq y}$ are a disjoint union of $\dim(\mathcal{Y}_{\geq y})$ -spherical spaces. In this case the functor $L_q(f^* \mathbb{Z})$ is non-zero only for objects y such that $\dim \mathcal{Y}_{\geq y} = q$. The E^2 -page of the spectral sequence 2.6.1 becomes

$$\begin{aligned} E_{p,q}^2 &= \operatorname{colim}_p L_q(f^* \mathbb{Z}) \cong \bigoplus_{\dim \mathcal{Y}_{\geq y} = q} H_p(\Sigma \mathcal{Y}_{< y}) \otimes H_q(f_{\geq y}) \quad \text{for } p > 0 \\ E_{p,0}^2 &= \operatorname{colim}_p H_0(f_{\geq -}). \end{aligned} \quad (2.6.2)$$

Moreover assume that $\mathcal{Y}_{< y}$ is $\dim(\mathcal{Y}_{< y})$ -spherical for all y . If $n = \dim \mathcal{Y}$ then the dimension of \mathcal{X} is also n since the supremum of the dimensions of the fibers is n . The spectral sequence

collapses in E^2 -page. The only non-zero terms are in degrees (p, q) such that $p + q = n$ or $q = 0$. Therefore there is a filtration

$$0 = F_{n+1} \subset F_n \subset \cdots \subset F_1 \subset F_0 = H_n(\mathcal{X})$$

where

$$F_q/F_{q+1} \cong \bigoplus_{\dim Y_{\geq y}=q} \tilde{H}_{n-q-1}(\mathcal{Y}_{<y}) \otimes \tilde{H}_q(f_{\geq y}) \quad \text{for } 1 \leq q \leq n$$

and

$$\operatorname{colim}_{\mathcal{Y}^{\text{op}}} H_0(f_{\geq -}) \cong \begin{cases} F_0/F_1 & \text{if } p = n \\ H_p(\mathcal{X}) & \text{if } 0 \leq p \leq n-1. \end{cases}$$

If all the fibers $f_{\geq y}$ are connected then this spectral sequence can be used to deduce [10, Theorem 9.1].

Theorem 2.7. [10] *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map of posets. If \mathcal{Y} is $(\dim \mathcal{Y})$ -spherical, $f_{\geq y}$ is $\dim(\mathcal{Y}_{\geq y})$ -spherical and $\mathcal{Y}_{<y}$ is $\dim(\mathcal{Y}_{<y})$ -spherical for all y then \mathcal{X} is $(\dim \mathcal{Y})$ -spherical.*

We will apply the spectral sequence 2.6.2 to the natural map

$$\epsilon : O_G \mathcal{F} \rightarrow \mathcal{F}$$

defined by $gA \mapsto A$. Let $G_{\geq A}$ denote the subgroup of G generated by the subgroups in the collection $\mathcal{F}_{\geq A}$. The fibers $\epsilon_{\geq A}$ can be identified with $O_G \mathcal{F}_{\geq A}$ whose connected components are in bijective correspondence with the coset representatives $G/G_{\geq A}$.

Theorem 2.8. *Assume that \mathcal{F} is $(\dim \mathcal{F})$ -spherical, $O_G \mathcal{F}_{\geq A}$ is a disjoint union of $\dim(\mathcal{F}_{\geq A})$ -spherical posets and $\mathcal{F}_{<A}$ is $\dim(\mathcal{F}_{<A})$ -spherical for all $A \in \mathcal{F}$. If for all $A \in \mathcal{F}$ non-maximal $G_{\geq A} = G$ then $O_G \mathcal{F}$ is $(\dim \mathcal{F})$ -spherical.*

Proof. If $\dim \mathcal{F} = 0$ that is every object is maximal then the space $O_G \mathcal{F}$ is a disjoint union of points. Assume that $\dim \mathcal{F} = n \geq 1$. Then there is at least one A with $G_{\geq A} = G$ which implies that $O_G \mathcal{F}$ is connected. We need to check that the derived colimits

$$\operatorname{colim}_{\mathcal{F}^{\text{op}}} H_0(f_{\geq -}) \cong \operatorname{colim}_{\mathcal{F}^{\text{op}}} \mathbb{Z}G/G_{\geq -}$$

vanish for $0 < p < n$ and gives \mathbb{Z} when $p = 0$. By the assumption the functor $A \mapsto \mathbb{Z}G/G_{\geq A}$ takes the value \mathbb{Z} for non-maximal objects. There is an exact sequence of functors

$$0 \rightarrow K \rightarrow \mathbb{Z}G/G_{\geq -} \rightarrow \mathbb{Z} \rightarrow 0 \quad (2.8.1)$$

induced by the augmentation map $\mathbb{Z}G/G_{\geq A} \rightarrow \mathbb{Z}$ for each object. The derived colimit of K is non-zero only at degree n which is given by

$$\operatorname{colim}_n K \cong \bigoplus_{i=1}^k H_n(\Sigma \mathcal{F}_{<M_i}) \otimes K(M_i)$$

where M_1, M_2, \dots, M_k are the maximal objects in \mathcal{F} . For the constant functor we have $\operatorname{colim}_i \mathbb{Z} \cong H_i(\mathcal{F})$. Then the result follows from the long exact sequence associated to 2.8.1. \square

3. SPHERICAL COSET POSETS

Extraspecial groups. An extraspecial p -group is an extension of the form

$$0 \rightarrow \mathbb{Z}/p \rightarrow E \xrightarrow{\pi} V \rightarrow 0$$

where the kernel is both the center and the commutator of the group [2, Chapter 8]. V is the Frattini quotient which is isomorphic to $(\mathbb{Z}/p)^{2n}$. Let $\mathcal{A}(G)$ denote the collection of abelian subgroups of a group G . Our main result is the following theorem.

Theorem 3.1. *The universal cover of $O_E\mathcal{A}(E)$ is a wedge of n -spheres.*

The commutator induces a bilinear form on V : $B(x, y) = [\pi^{-1}(x), \pi^{-1}(y)]$. Abelian subgroups of E containing the center are in one to one correspondence with isotropic subspaces of V i.e. $A \subset V$ such that $B(x, y) = 0$ for all $x, y \in A$. Let $\mathcal{I}(V)$ denote the collection of isotropic subspaces of V . There is a map $\mathcal{A}(E) \rightarrow \mathcal{I}(V)$ induced by π . There is also an induced map between the cosets $O_E\mathcal{A}(E) \rightarrow O_V\mathcal{I}(V)$. We start with an elementary observation which can be proved by showing that the fibers are contractible.

Proposition 3.2. *The induced map $O_E\mathcal{A}(E) \rightarrow O_V\mathcal{I}(V)$ is a homotopy equivalence.*

The dimension of $O_V\mathcal{I}(V)$ is the length of the longest chain of isotropic subspaces. It is a well-known fact that the dimensions of all the maximal isotropic subspaces are equal to half the dimension of V , see [2, Chapter 7]. Therefore we have $\dim(O_V\mathcal{I}(V)) = n$. Let us consider the smallest case $n = 1$. In this case E is either D_8 or Q_8 and the homotopy type of $O_E\mathcal{A}(E)$ is studied in [1, §7]. It turns out that in both cases $O_E\mathcal{A}(E)$ is a wedge of circles.

In the rest of this section we assume that $n \geq 2$. To describe the fundamental group we introduce the Heisenberg group $H(V)$ associated to a vector space with a bilinear form (V, B) . This group is the set $V \times \mathbb{Z}/p$ with the multiplication rule

$$(v_1, t_1)(v_2, t_2) = (v_1 + v_2, B(v_1, v_2) + t_1 + t_2).$$

There is a natural projection $H(V) \rightarrow V$. An isotropic subspace A can be identified with an abelian subgroup of $H(V)$ by the map $a \mapsto (a, 0)$.

Proposition 3.3. [8, 9] *The natural map $O_{H(V)}\mathcal{I}(V) \rightarrow O_V\mathcal{I}(V)$ is the universal covering map where $\dim V \geq 4$.*

Proof. The $p = 2$ case is studied in [8]. The general case can be deduced from [9]. There is a map of fibrations

$$\begin{array}{ccccc} O_E\mathcal{A}(E) & \longrightarrow & \operatorname{hocolim}_{\mathcal{A}(E)} B & \longrightarrow & BE \\ \downarrow & & \downarrow & & \downarrow \\ O_V\mathcal{I}(V) & \longrightarrow & \operatorname{hocolim}_{\mathcal{I}(V)} B & \longrightarrow & BV \end{array}$$

where $B : \mathcal{A}(E) \rightarrow \mathbf{S}$ is the classifying space functor, similarly defined on $\mathcal{I}(V)$. The fundamental group of $\operatorname{hocolim}_{\mathcal{A}(E)} B$ is isomorphic to the colimit of the groups in the collection $\mathcal{A}(E)$. By [9, Theorem 3.8] this colimit is isomorphic to the kernel $D(E)$ of the multiplication map $E \times E \rightarrow E/[E, E]$. One can check that this gives a diagram of groups

$$\begin{array}{ccccc} \mathbb{Z}/p & \hookrightarrow & D(E) & \twoheadrightarrow & E \\ \parallel & & \downarrow & & \downarrow \\ \mathbb{Z}/p & \hookrightarrow & H(V) & \twoheadrightarrow & V \end{array}$$

which implies that $\pi_1(\operatorname{hocolim}_{\mathcal{I}(V)} B) \cong H(V)$. The fibration

$$O_{H(V)}\mathcal{I}(V) \rightarrow \operatorname{hocolim}_{\mathcal{I}(V)} B \rightarrow B(H(V))$$

implies that $O_{H(V)}\mathcal{I}(V)$ is simply connected. □

In the rest of this section we will study $O_{H(V)}\mathcal{I}(V)$ and show that it is a wedge of n -spheres. Since its dimension is equal to the dimension of $O_V\mathcal{I}(V)$ it suffices to show that it is n -spherical. As a preparation we study variations of the Tits complex associated to the general linear group.

Poset of proper subspaces. Let $\mathcal{T}(V)$ denote the poset of proper subspaces of V , and $\mathcal{T}^\circ(V) = \mathcal{T}(V) - \{0\}$. The geometric realization of the latter poset is the Tits complex associated to the general linear group defined over \mathbb{F}_p . Homotopy type of Tits complexes is described by the celebrated Solomon–Tits theorem.

Proposition 3.4. [11] $\mathcal{T}^\circ(V)$ is $(\dim V - 2)$ -spherical.

In his work on discrete series representations of the classical groups Lusztig studied the poset $O_V\mathcal{T}(V)$. In [7, Theorem 1.9] he proves a homological version of the following.

Proposition 3.5. $O_V\mathcal{T}(V)$ is $(\dim V - 1)$ -spherical.

This result is a special case of [4, Proposition 11] where it is extended to cosets of proper subgroups of a solvable group.

Let W be a subspace of V . Assume $\operatorname{codim} W = 1$ and take $v \notin W$. We define a map

$$\theta_v : \mathcal{T}(V)^W \rightarrow O_W\mathcal{T}(W)$$

where $\theta_v(A) = (-v + A) \cap W$. Note that this coincides with the composition of $\theta_W(-v)$ with i_W after identifying $\mathcal{T}(V)_{\cap W}$ with $\mathcal{T}(W)$.

Proposition 3.6. The map $\theta_v : \mathcal{T}(V)^W \rightarrow O_W\mathcal{T}(W)$ is a homotopy equivalence. In particular $\mathcal{T}(V)^W$ is $(\dim V - 2)$ -spherical.

Proof. The homotopy inverse is given by the map

$$s_v : O_W \mathcal{T}(W) \rightarrow \mathcal{T}(V)^W$$

defined by $s_v(w + A) = \langle v + w, A \rangle$. One checks that $\theta_v s_v(w + A) \supset w + A$, and $s_v \theta_v(A) \subset A$. \square

For $U \subset V$ and a poset \mathcal{F} of subspaces define $\mathcal{F}_U = \{A \in \mathcal{F} \mid A \cap U = 0\}$. We will denote the intersection $\mathcal{F}^W \cap \mathcal{F}_U$ simply by \mathcal{F}_U^W . There is a commutative diagram

$$\begin{array}{ccc} \mathcal{T}(V)^W & \xrightarrow{\theta_v} & O_W \mathcal{T}(W) \\ \uparrow & & \uparrow \\ \mathcal{T}(V)_U^W & \xrightarrow{\theta'_v} & O_W \mathcal{T}(W)_U \end{array} \quad (3.6.1)$$

where θ'_v is the restriction of θ_v . The map θ'_v is also a homotopy equivalence whose homotopy inverse is given by the restriction of s_v to $O_W \mathcal{T}(W)_U$. Moreover they are spherical.

Theorem 3.7. *Let $U \subset W$ and $\text{codim } W = \dim U = 1$. Then $O_W \mathcal{T}(W)_U \simeq \mathcal{T}(V)_U^W$ is $(\dim V - 2)$ -spherical.*

Proof. We will show $O_W \mathcal{T}(W)_U$ is $(\dim W - 1)$ -spherical by induction on the dimension of W . Let $L \supset U$ be a codimension one subspace of W . We decompose the space with respect to L and use 2.6. Note that $\mathcal{T}(W)_U$ is L -stable. Then by 2.2 $O_L \mathcal{T}(W)_U \rightarrow O_L(\mathcal{T}(W)_U)_{\cap L}$ is a weak equivalence, and the latter can be identified with $O_L \mathcal{T}(L)_U$. Therefore it suffices to construct a section s satisfying the properties in 2.5 and show that $O_W \mathcal{T}(W)_U^L$ and $O_L \mathcal{T}(L)_U$ are $(\dim W - 2)$ -spherical. The latter satisfies this by induction. Let us focus on $O_W \mathcal{T}(W)_U^L$ and the construction of s . Consider the inclusion map

$$\theta_L : O_W \mathcal{T}(W)_U^L \rightarrow O_L \mathcal{T}(L)_U$$

and choose an element $v \notin L$. As in ?? the map

$$\theta'_v : \mathcal{T}(W)_U^L \rightarrow O_L \mathcal{T}(L)_U$$

has a homotopy inverse s_v defined by $l + A \mapsto \langle v + l, A \rangle$. Check also that $\theta'_v s_v(l + A) = \langle v + l, A \rangle \cap L \supset 0$ that is $\theta'_v s_v \simeq 0$. Therefore s_v satisfies the conditions in order to apply 2.6. Next we determine the homotopy type of $O_W \mathcal{T}(W)_U^L$ by applying 2.8 to the natural map $\epsilon : O_W \mathcal{T}(W)_U^L \rightarrow \mathcal{T}(W)_U^L$ defined by $gA \mapsto A$. We have

$$\begin{aligned} (\mathcal{T}(W)_U^L)_{<A} &= (\mathcal{T}(W)^L)_{<A} = \mathcal{T}(A)^{L \cap A} \\ \epsilon_{\geq A} &= O_W(\mathcal{T}(W)_U^L)_{\geq A} = O_W(\mathcal{T}(W)_U)_{\geq A} \simeq O_{W/A} \mathcal{T}(W/A)_{U A/A} \end{aligned}$$

where the second space is connected if A is not maximal. The homotopy type of the first one is given by 3.6 and the second one is determined by induction. Therefore by 2.8 $O_W \mathcal{T}(W)_U^L$ is $(\dim W - 2)$ -spherical. Now 2.6 gives the desired result. \square

Poset of isotropic subspaces. Let V be a vector space over \mathbb{F}_p with a non-degenerate antisymmetric bilinear form B i.e $B(x, y) = -B(y, x)$. The complement of A is defined by $A^\perp = \{v \mid B(v, a) = 0 \ \forall \ a \in A\}$. A subspace is called isotropic if $A \subset A^\perp$. The collection of isotropic subspaces will be denoted by $\mathcal{I}(V)$. We fix a symplectic basis $b = \{x_1, x_2, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ of V such that $B(x_i, \bar{x}_j) = 1$ when $i = j$ and zero otherwise.

Let us fix $x = x_n$ and $\bar{x} = \bar{x}_n$ for the rest. The quotient $x^\perp / \langle x \rangle$ can be identified with the subspace Z spanned by $b - \{x, \bar{x}\}$. Let $j : x^\perp / \langle x \rangle \rightarrow Z \subset x^\perp$ denote this identification. There is a commutative diagram

$$\begin{array}{ccc} x^\perp & \xrightarrow{k} & x^\perp \\ \downarrow p & \nearrow j & \\ x^\perp / \langle x \rangle & & \end{array}$$

where $k(u) = u - B(u, \bar{x})x$. There is a corresponding commutative diagram

$$\begin{array}{ccc} H(x^\perp) & \xrightarrow{k} & H(x^\perp) \\ \downarrow p & \nearrow j & \\ H(x^\perp / \langle x \rangle) & & \end{array} \quad (3.7.1)$$

of Heisenberg groups. Let us write the element $(v, 0)$ in $H(V)$ as (v) to simplify the notation. Note that for an isotropic subspace A the group $H(A^\perp)$ can be identified with the centralizer of A in $H(V)$.

Proof of the main theorem. Now we are ready for the proof of Theorem 3.1. By 3.2 and 3.3 it reduced to proving that $O_{H(V)}\mathcal{I}(V)$ is n -spherical where $2n$ is the dimension of V . We will apply 2.6 with $H = H(x^\perp)$. Note that $H(x^\perp)$ is a normal subgroup of $H(V)$ whose quotient is isomorphic to a cyclic group of prime order. In order to apply 2.6 we have two main objectives: First construct a homotopy section s , and second show that $O_V\mathcal{I}(V)^{H(x^\perp)}$ is $(n-1)$ -spherical.

Let us start with some elementary observations. The quotient map $p : x^\perp \rightarrow x^\perp / \langle x \rangle$ induces a weak equivalence

$$p : O_{H(x^\perp)}\mathcal{I}(x^\perp) \rightarrow O_{H(x^\perp / \langle x \rangle)}\mathcal{I}(x^\perp / \langle x \rangle)$$

since it can be shown that the fibers canonically contract to a terminal object. Here $H(x^\perp) / \langle x \rangle$ is identified with $H(x^\perp / \langle x \rangle)$. Identifying $\mathcal{I}(V)_{\cap H(x^\perp)}$ with $\mathcal{I}(x^\perp)$ the map in 2.2.1 can be written as

$$i_{H(x^\perp)} : O_{H(x^\perp)}\mathcal{I}(V) \rightarrow O_{H(x^\perp)}\mathcal{I}(x^\perp)$$

and it is a weak equivalence by 2.2. Let ϕ denote the composition $pi_{H(x^\perp)}$.

Recall the decomposition given in 2.2.3 where we take $H = H(x^\perp)$. There is an inclusion

$$\theta_{H(x^\perp)} : \mathcal{I}(V)^{H(x^\perp)} \rightarrow O_{H(x^\perp)}\mathcal{I}(V).$$

The left cosets of $H(x^\perp)$ in the group $H(V)$ are translates under the elements $(\bar{x}), (\bar{x})^2, \dots, (\bar{x})^{p-1}$. We take $g = (\bar{x})$ in 2.5. Our first objective is to construct a homotopy section s for the map $\theta_{H(x^\perp)}(\bar{x})^{-1}$ satisfying the requirements of 2.5. It will suffice to construct an inverse \bar{s} of the map

$$\bar{\theta} = \bar{\theta}_{H(x^\perp)}(\bar{x})^{-1} : \mathcal{I}(V)^{H(x^\perp)} \rightarrow O_{H(x^\perp/\langle x \rangle)} \mathcal{I}(x^\perp/\langle x \rangle) \quad (3.7.2)$$

where $\bar{\theta}_{H(x^\perp)} = \phi \theta_{H(x^\perp)}$, and this section should satisfy $\bar{\theta}_{H(x^\perp)} \bar{s} \simeq 0$. To see this we define $s = \bar{s} \phi$. If φ is a homotopy inverse of ϕ then multiplying $\bar{\theta} \bar{s} = 1$ by φ on the left and by ϕ on the right we get $\theta_{H(x^\perp)}(\bar{x})^{-1} s \simeq 1$. Moreover $\theta_{H(x^\perp)} s \simeq 0$ since $\phi \theta_{H(x^\perp)} s \varphi \simeq \bar{\theta}_{H(x^\perp)} \bar{s} \simeq 0$.

We proceed with constructing \bar{s} . The diagram ?? will be useful. We define a map of posets

$$\bar{s} : O_{H(x^\perp/\langle x \rangle)} \mathcal{I}(x^\perp/\langle x \rangle) \rightarrow \mathcal{I}^{H(x^\perp)} \quad (3.7.3)$$

which sends a coset $(w, t)A \subset j(H(x^\perp/\langle x \rangle))$ to the subgroup

$$\langle (\bar{x} + tx + w), (B(w, a)x + a) \mid a \in A \rangle.$$

Let us check that this definition gives an abelian group. Note that the commutator $[(v_1), (v_2)]$ of two elements in $H(V)$ is simply $2B(v_1, v_2)$ regarded as an element of $H(V)$. Therefore \bar{s} applied to a coset gives an abelian subgroup since $B(\bar{x} + tx + w, B(w, a)x + a) = B(w, a)B(\bar{x}, x) + B(w, a) = 0$.

Lemma 3.8. *The map \bar{s} is the inverse of $\bar{\theta}$ as defined in 3.7.2 and satisfies $\bar{\theta}_{H(x^\perp)} \bar{s} \simeq 0$.*

Proof. Let A be an isotropic subspace in V such that $A + x^\perp = V$. We can regard A as a subgroup of $H(V)$ by mapping $a \mapsto (a, 0)$. We have $\bar{x} + u \in A$ for some $u \in x^\perp$ since A and \bar{x} span the whole space V . Then A is spanned by the vector $\bar{x} + u$ and some vectors in x^\perp that is $A = \langle \bar{x} + u, A \cap x^\perp \rangle$. Recall that $\bar{\theta} = \bar{\theta}_H g^{-1}$ sends a space A to the quotient $((g^{-1}A) \cap H)/\langle x \rangle$ where $H = H(x^\perp)$ and $g = (\bar{x})$. Although the function $\bar{\theta}$ is defined on subspaces it will be convenient to think that it is defined on vectors. This way we get $\bar{\theta}(v) = p((\bar{x})^{-1}(v)) = p(v - \bar{x}, B(v, \bar{x}))$ for (v) in $H(V)$ where p is the canonical projection $x^\perp \rightarrow x^\perp/\langle x \rangle$. The composition $j\bar{\theta}$ sends (v) to $(k(v - \bar{x}), B(v, \bar{x}))$ by the commutativity of ??. In particular we have $j\bar{\theta}(\bar{x} + u) = (k(u), B(u, \bar{x}))$. More generally for $a \in A \cap x^\perp$ we compute

$$\begin{aligned} j\bar{\theta}(\bar{x} + u + a) &= (k(u + a), B(u + a, \bar{x})) \\ &= (k(u), B(u, \bar{x}))(k(a), B(a, \bar{x}) + B(a, u)) \\ &= (k(u), B(u, \bar{x}))(k(a), 0) \end{aligned}$$

since a and $\bar{x} + u$ are elements of an isotropic space A i.e. $B(a, \bar{x} + u) = 0$. Therefore the composition $j\bar{\theta}$ maps

$$A = \langle \bar{x} + u, A \cap x^\perp \rangle \mapsto (k(u), B(u, \bar{x}))k(A)$$

Applying \bar{s} to this coset gives a subgroup generated by $(\bar{x} + B(u, \bar{x})x + k(u)) = (\bar{x} + u)$ and

$$(B(k(u), k(a))x + k(a)) = (B(u, a)x + k(a)) = (B(a, \bar{x})x + k(a)) = (a)$$

where we used the fact that k is defined by $k(u) = u - B(u, \bar{x})x$ and respects the bilinear form. Therefore

$$\bar{s}\bar{\theta}(A) = \langle (\bar{x} + u), (a) \mid a \in A \cap x^\perp \rangle = A$$

for all isotropic A . On the other hand let $(w, t)A$ be a coset in the image of j . Each element $(w, t)(a, 0)$ can be written as $(w + a, t + B(w, a))$. After applying \bar{s} consider the element $(\bar{x} + tx + w + B(w, a)x + a)$ obtained by the product of two generators. To see what $\bar{\theta}$ does to $\bar{s}((w, t)A)$ it suffices to check its effect on this element

$$\begin{aligned} (-\bar{x})(\bar{x} + tx + w + B(w, a)x + a) &= (tx + w + B(w, a)x + a, t + B(w, a)) \\ &\equiv (w + a, t + B(w, a)) \pmod{(x)} \end{aligned}$$

which implies that $\bar{\theta}\bar{s}$ is the identity. The second statement follows from $\bar{\theta}_{H(x^\perp)}\bar{s}((w, t)A) \supset 0$ since these maps are defined by intersecting with the group $H(x^\perp)$ and taking the quotient by $\langle x \rangle$. Therefore this composition contracts to the constant map at the zero subspace. \square

We now turn to our second goal concerning $O_{H(V)}\mathcal{I}(V)^{H(x^\perp)}$. Restricting the domains of the maps $i = i_{H(x^\perp)}$ and p we obtain the following maps

$$O_{H(V)}\mathcal{I}(V)^{H(x^\perp)} \xrightarrow{i} O_{H(x^\perp)}\mathcal{I}(x^\perp)_{\langle x \rangle} \xrightarrow{p} O_{H(x^\perp/\langle x \rangle)}\mathcal{I}(x^\perp/\langle x \rangle). \quad (3.8.1)$$

Let us consider the fibers of these maps. Since both maps are equivariant with respect to the natural action of $H(V)$ it suffices to consider the fiber over a coset of the form B for some subgroup. Other cosets αB are isomorphic to this one under the action of $H(V)$.

Lemma 3.9. *The fiber $p_{\leq B}$ is $(\dim B)$ -spherical.*

Proof. Lifting B to an isotropic subspace $\tilde{B} = \langle j(B), x \rangle$ in x^\perp we have

$$p_{\leq B} = O_{\tilde{B}}\mathcal{T}(\tilde{B})_{\langle x \rangle} \simeq \mathcal{T}(\tilde{B} \oplus \mathbb{F}_p)_{\langle x \rangle}^{\tilde{B}}$$

which is $(\dim(\tilde{B} \oplus \mathbb{F}_p) - 2)$ -spherical by 3.7. \square

Lemma 3.10. *The fiber $i_{\geq B}$ is $(n - \dim B - 2)$ -spherical.*

Proof. We have

$$i_{\geq B} = (\mathcal{I}(V)^{H(x^\perp)})_{\geq B} \cong \mathcal{I}(B^\perp/B)^{H(B^\perp \cap x^\perp)/B}$$

where the last one is isomorphic to $O_{H(B^\perp \cap x^\perp/\langle B, x \rangle)}\mathcal{I}(B^\perp \cap x^\perp/\langle B, x \rangle)$ by 3.8. The dimension of a maximal isotropic subspace in $B^\perp \cap x^\perp/\langle B, x \rangle$ is $n - \dim B - 2$. \square

Now we can finish the proof of the main theorem by collecting the results obtained so far.

Proof of Theorem 3.1. We reduced the proof to showing that $O_{H(V)}\mathcal{I}(V)$ is n -spherical where n is half the dimension of V . We will do induction on n . The cases $n = 1, 2$ both follow from 3.3. For $n > 2$ it remains to prove that

$$\mathcal{X} = O_{H(V)}\mathcal{I}(V)^{H(x^\perp)}$$

is $(n - 1)$ -spherical. We will use 3.9 and 3.10. First consider the map

$$p : \mathcal{X}' \rightarrow \mathcal{Y}'$$

in ???. By the induction assumption \mathcal{Y}' is $(n - 1)$ -spherical. The fiber $p_{\leq B}$ is $(\dim B)$ -spherical by 3.9, and $\mathcal{Y}'_{>B} = O_{H(B^\perp/B)}\mathcal{I}(B^\perp/B)$ is $(n - 1 - \dim B)$ -spherical by inspection. Then \mathcal{X}' is $(n - 1)$ -spherical by 2.7 applied to the opposite of the map $\mathcal{X}' \rightarrow \mathcal{Y}'$. We turn to the other map in ??, and denote it simply by

$$i : \mathcal{X} \rightarrow \mathcal{Y}$$

where $\mathcal{Y} = \mathcal{X}'$. In this case $\mathcal{Y}_{<B} = O_B\mathcal{T}(B)_{\langle x \rangle}$ is $(\dim B - 1)$ -spherical by 3.6 and $i_{\geq B}$ is $(n - \dim B - 2)$ -spherical by 3.10. Therefore \mathcal{X} is $(n - 1)$ -spherical by 2.7. \square

REFERENCES

- [1] Alejandro Adem, Frederick R. Cohen, and Enrique Torres Giese. Commuting elements, simplicial spaces and filtrations of classifying spaces. *Math. Proc. Cambridge Philos. Soc.*, 152(1):91–114, 2012.
- [2] Michael Aschbacher. *Finite group theory*, volume 10 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.
- [3] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.
- [4] Kenneth S. Brown. The coset poset and probabilistic zeta function of a finite group. *J. Algebra*, 225(2):989–1012, 2000.
- [5] William G. Dwyer and Hans-Werner Henn. *Homotopy theoretic methods in group cohomology*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2001.
- [6] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
- [7] George Lusztig. *The discrete series of GL_n over a finite field*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 81.
- [8] Cihan Okay. Homotopy colimits of classifying spaces of abelian subgroups of a finite group. *Algebr. Geom. Topol.*, 14(4):2223–2257, 2014.
- [9] Cihan Okay. Colimits of abelian groups. *J. Algebra*, 443:1–12, 2015.
- [10] Daniel Quillen. Homotopy properties of the poset of nontrivial p -subgroups of a group. *Adv. in Math.*, 28(2):101–128, 1978.
- [11] Louis Solomon. The Steinberg character of a finite group with BN -pair. In *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*, pages 213–221. Benjamin, New York, 1969.
- [12] Karen Vogtmann. Spherical posets and homology stability for $O_{n,n}$. *Topology*, 20(2):119–132, 1981.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON ON N6A 5B7

E-mail address: cokay@uwo.ca